Chain Rule:

Multi variable chain Rule:

$z = f(x,y)$, $\quad x = x(t)$

$\qquad\qquad\qquad\quad y = y(t)$

$z = f(x(t), y(t))$

$$\frac{dz}{dt} = \frac{df}{dt} = \frac{df}{dx}\cdot\frac{dx}{dt} + \frac{df}{dy}\cdot\frac{dy}{dt}$$

$$= f_x(x(t), y(t) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t)$$

Ex) $r(t) = (\cos t, \sin t)$, $f(x,y) = x^2 - y^2$

Find $\frac{df}{dt} = 2x(-\sin t) + (-2y)(\cos t)$

$f_x = 2x$

$f_y = -2y$

$= 2(\cos t)(-\sin t) - 2\sin t \cos t$

$= -2\sin t \cos t - 2\sin t \cos t$

$= -4\sin t \cos t = -2\sin 2t$

MCR in Three variables:

$r(t) = (x(t), y(t), z(t))$

$w = f(x,y,z)$

$\Big\} \; w = f(r(t))$

$dw \; = \; df \; = \; df \; dx \; + \; df \; dy \; + \; df \; dz$

$dt \quad dt \quad dx \; dt \quad dy \; dt \quad dz \; dt$

Note: This is a dot product

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot r'(t)$$

we will see the vector as a gradient f soon.

Ex) $xy + z^3x - 2yz = 0$ (hint y is the constant)

find $\frac{dz}{dx}$ at the point $(1,1,1)$

$$y + 3z^2\frac{dz}{dx}x + z^3 - 2y\frac{dz}{dx} = 0$$

$$3z^2x\frac{dz}{dx} - 2y\frac{dz}{dx} = -y - z^3$$

$$\frac{dz}{dx} = \frac{-y-z^3}{3z^2x - 2y} = \left.\frac{y+z^3}{2y-3z^2x}\right|_{(1,1,1)} = -2$$

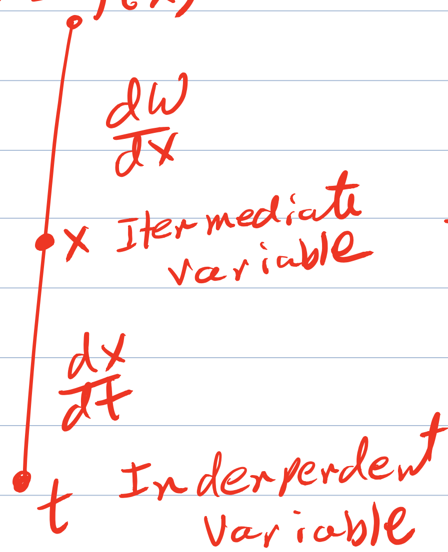
Find $\frac{dw}{dt}$ if $w = xy$ with respect to t

along the path $x = \cos t$, $y = \sin t$, where $t = \pi/2$?

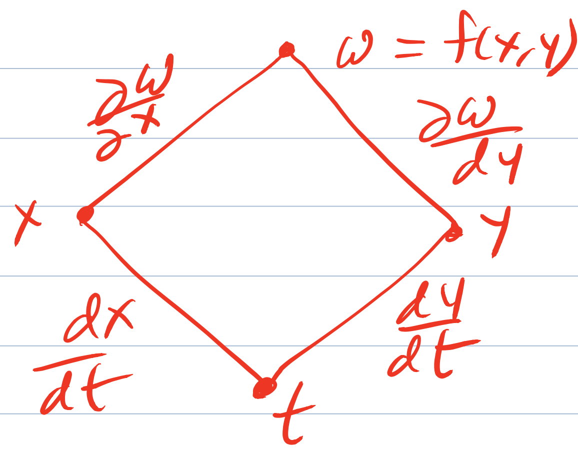
$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

to find $\frac{dw}{dt}$

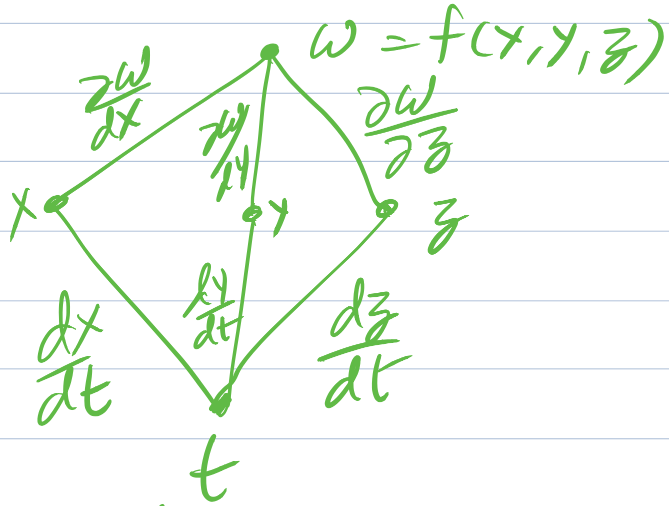
$w = f(x)$



$$\Rightarrow \frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}$$



$$\hookrightarrow \frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$



$$\hookrightarrow \frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

lets get back to the problem

$$= \frac{\partial}{\partial}(xy)\cdot\frac{d}{dt}\cos t + \frac{\partial}{\partial y}(xy)\cdot\frac{d}{dt}(\sin t)$$

$$= y(-\sin t) + x(\cos t)$$

$$= \sin t(-\sin t) + \cos t(\cos t)$$

$$= -\sin^2 t + \cos^2 t = \cos 2t \Big|_{t=\frac{\pi}{2}}$$

$$= -1$$

Ex) Find $\frac{dw}{dt}$, if

dt,

$w = xy + z$, $x = \cos t$, $y = \sin t$, $z = t$

+ What is the derivatives value at $t=0$?

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= y(-\sin t) + (x)(\cos t) + (1)(1)$$

$$= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1$$

$$= \cos 2t + 1 \Big|_{t=0} = 2$$

Chain Rule for two independent variables + three intermediate variables:

Let $w = f(x, y, z)$, $x = g(r, s)$

$y = h(r, s)$

$z = k(r, s)$

Then $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{dz}{\partial r}$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

Ex) Express $\frac{\partial w}{\partial r}$ & $\frac{\partial w}{\partial s}$ in terms of r & s if $w = x + 2y + z^2$,

$x = \frac{r}{s}$, $y = r^2 + \ln s$, $z = 2r$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$(1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2)$$

$$= \frac{1}{s} + 4r + 4z = \frac{1}{s} + 4r + 8r = \frac{1}{s} + 12r$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1)\left(-\frac{r}{s^2}\right) + 2\left(\frac{1}{s}\right) + 2z(0)$$

$$= \frac{2}{s} - \frac{r}{s^2}$$